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RH WAVELET BASES TO APPROXIMATE SOLUTION OF NONLINEAR VOLTERRA - FREDHOLM - HAMMERSTEIN INTEGRAL EQUATIONS WITH ERROR ANALYSIS

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Abstract. In this paper, we present a method for calculated the numerical approximation of nonlinear Fredholm - Volterra Hammerstein integral equation, which uses the properties of rationalized Haar wavelets. The main tool for error analysis is the Banach fixed point theorem. An upper bound for the error was obtained and the order of convergence is analyzed. An algorithm is presented to compute and illustrate the solutions for some numerical examples.

Keywords: nonlinear integral equation, rationalized haar wavelet, operational matrix, fixed point theorem, error analysis.

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1. Introduction and preliminaries

Due to this, many numerical methods have been developed for finding the solutions of integral equations. The use of wavelets has come to prominence during the last two decades. Wavelets can be used as analytical tools for signal processing, numerical analysis and mathematical modeling. The early works concerning wavelets were in the 1980s by Morlet , Grossmann , Meyer, Mallat and others. But in fact, it was the paper of Daubechies [6] in 1988 that caught the attention of the applied mathematics communities in signal processing, and numerical analysis. Most of the early works are discussed in[5,12] and [6,7,16]. The goal of the most modern wavelet researches is to create a set of basis functions and transform them, which yields an informative and useful description of a function or signal.Various types of wavelets have been applied for numerical solution of different kinds of integral equations.

These include Haar, Legendre, trigonometric, CAS, Chebyshev, and Coifman wavelets. Lepik and Tamme in[10] have applied Haar wavelets to nonlinear Fredholm integral equations, but their method involves approximation of certain integrals. The orthogonal set of Haar functions is a group of square waves with magnitude of $+2^{\frac{i}{2}}, -2^{\frac{i}{2}}$ and 0, for any $i = 0,1, \ldots$[13,14] . Lynch and Reis [11] have rationalized the Haar transform by deleting the irrational numbers and introducing the integral powers of two. This modification results in what is called the rationalized Haar (RH) transform. The RH transform preserves all the properties of the original Haar transform and can be efficiently implemented using digital pipeline architecture [15]. The corresponding functions are known as RH functions. The RH functions are composition of only three amplitude +1,-1 and 0. The aim of this work is to present a numerical method for approximating the solution of nonlinear Fredholm - Hammerstein integral equations of the second kind as follows:

$$u(x) = f(x) + \alpha \int_0^1 K_1(x,t) W_1(t, u(t)) dt, \tag{1}$$

$$u(x) = f(x) + \beta \int_0^1 K_2(x,t) W_2(t, u(t)) dt, \tag{2}$$

where $f: [0,1] \to R$, and $W_1, W_2: [0,1] \times R \to R$, $\alpha, \beta \in R$, and $K_1, K_2: [0,1]^2 \to R$ are assumed to be known continuous functions, and the unknown function to be determined is $u: [0,1] \to R$, and $u \in X = C([0,1])$. Also we assume that:

1. $f(t) \in C([0,1])$.
2. $K_1, K_2 \in C([0,1]^2)$, then there exists $M_1, M_2 \geq 0$ such that $|K_i(x,t)| \leq M_i$ for $i = 1,2$.
3. $W_1, W_2: [0,1] \times R \to R$, is a continuous function such thatthere exists $L_1, L_2 > 0$ where Wsatisfy a global Lipschitz condition for $x \in [0,1]$ and for all$y, z \in R$, namely $|W_i(x,y) - W_i(x,z)| \leq L_i|y - z|$,for $i = 1,2$.
4. $M_1 L_1 < 1$, $M_2 L_2 < 1$.

There exist several numerical methods for Fredholm-Hammerstein integral equations. The classical successive approximations method has been introduced in [17]. A variation of the Nystrom method was presented in [9]. A collocation-type method was developed in [8], and cubic spline interpolation is proposed in [4] . Equation (1) appears in reformulation of two - point BVP with a certain non-linear boundary condition [1].The numerical solution of nonlinear one-dimensional Fredholm - Hammerstein integral equations using a basis of Haar functions was considered by Razzaghi and Ordokhani in [13,14]. The numerical results presented in that paper show a fast convergence of their method, when applied to integral equations. The integral operator $T: (X, ||.||_\infty) \to (X, ||.||_\infty)$ is also defined as

$$T\big(u(x)\big) = f(x) + \alpha \int_0^1 K_1(x,t) W_1\big(t, u(t)\big) dt\,, \;\; t \in [0,1] \tag{3}$$

$$T\big(u(x)\big) = f(x) + \alpha \int_0^1 K_2(x,t) W_2(t, u(t)) dt\,, \;\; t \in [0,1]. \tag{4}$$

For all $y_1, y_2 \in C([0,1])$, we have $||Ty_1 - Ty_2|| \leq \widetilde{M}||y_1 - y_2||$, where $\widetilde{M} = ML$. Thus the Banach fixed point theorem guarantees that under certain assumptions [1], T has a unique fixed point; that is, the integral equation (1) has exactly one solution. Moreover, $u = lim_{n\to\infty} T^n\,(u_0)$, where u_0, is any continuous function on [0,1]. Since, in general it is not possible to calculate u explicitly from the sequence of functions $\{T^n(u)\}_{n\in N}$, we define a new sequence of functions, denoted by $\{u_i\}_{i\in N}$,obtained recursively using RH basis. More concretely, we could get u_{i+1} from u_i approximating $T(u_h)$ by means of the sequence of projections of such RH basis.

2. Properties of the rationalized Haar functions

Definition 1. The RH wavelet is the function defined on the real line R as follows:

$$H(t)=\begin{cases}1 & 0<t\le \frac{1}{2},\\ -1 & \frac{1}{2}<t<1,\\ 0 & otherwise.\end{cases}$$

The RH functions $h_l(t)$, for any $n=1,2,...$ where $l=2^i+j$, with $i=0,1,...$ and $j=0,1,\dots,2^i-1$, are defined by $h_l(t)=H(2^i t-j)|_{[0,1]}$. That is:

$$H\left(2^{\mathrm{i}}\,\mathrm{t}-\mathrm{j}\right)=\begin{cases}1 & \mathrm{j}.2^{-\mathrm{i}}<t\le(\mathrm{j}+\frac{1}{2})2^{-\mathrm{i}},\\ -1 & (\mathrm{j}+\frac{1}{2})2^{-\mathrm{i}}<t<(\mathrm{j}+1)2^{-\mathrm{i}},\\ 0 & otherwise.\end{cases}$$

Also, we define $h_0(t)=1$, for all $t\in[0,1)$,and integer $2^i, i=0,1,...$, indicates the level of the wavelet and $j=0,1,\dots,2^i-1$ is the translation parameter. Note that the basic multiplication properties of RH functions are as follows:

$$h_0(t)h_q(t)=h_q(t) for q\in Z^+\cup(0),$$

and for $0<l<q$, we have

$$\mathrm{h_l(t)h_q(t)}=\begin{cases}\mathrm{h_q(t)if\ h_q\ occurs\ during\ the\ positive\ half\ wave\ of\ h_l},\\ \mathrm{-h_q(t)if\ h_q\ occurs\ during\ the\ negative\ half\ wave\ of\ h_l},\\ 0 \qquad otherwise.\end{cases}$$

Also, it can be shown that the sequence$\{h_n\}_{n=0}^{\infty}$ is a complete orthogonal system in $L^2[0,1]$ .Note that the orthogonality property is :

$$\langle \mathrm{h_l(t),h_q(t)}\rangle=\int_0^1 \mathrm{h_l(t)h_q(t)dt}=\begin{cases}2^{-\mathrm{i}} & \mathrm{l=q=2^i+j},\\ 1 & \mathrm{l=q=0},\\ 0 & \mathrm{l\ne q}.\end{cases}$$

where $i\in Z^+\cup(0)$, and $j=0,1,\dots,2^i-1$.And for $f\in C[0,1]$,the series $\sum_n 2^j\langle f,h_l\rangle h_l$, converges uniformly to f, (see e.g.. [18]), where

$$\langle f,h_l\rangle=\int_0^1 f(t)h_l(t)dt.$$

Thus the function $f(x)$ in $L^2([0,1])$ can be expanded with finite terms of RH functions as

$$f(x)=\sum_{l=0}^{m-1} f_l h_l(x)=\boldsymbol{f}^T\boldsymbol{h}(x),$$

where $m=2^{\alpha+1}$ that $\alpha=0,1,\dots$, and the RH function coefficients f_l are given by:

$$f_l=\frac{\langle f(x),h_l(x)\rangle}{\langle h_l(x),h_l(x)\rangle}. \tag{5}$$

That vectors f and h are defined by $f=[f_0,f_1,\dots,f_{m-1}]^T$ and $h=[h_0(x),h_1(x),\dots,h_{m-1}(x)]^T$,and the integral $h_l(t)$ is given by

$$\int_0^1 h_l(t)dt=\begin{cases}1 & \mathrm{l}=0,\\ 0 & \mathrm{l}\neq 0.\end{cases} \tag{6}$$

Also we have:

$$\int_0^x h(s)\,ds=Ph(x), \tag{7}$$

where P is a $m\times m$ operational matrix for integration and is defined by

$$P_{m\times m}=\frac{1}{2m}\begin{pmatrix}2mP_{\frac{m}{2}\times\frac{m}{2}} & -\widehat{\Phi}_{\frac{m}{2}\times\frac{m}{2}}\\ \widehat{\Phi}^{-1}{}_{\frac{m}{2}\times\frac{m}{2}} & 0\end{pmatrix},$$

wherein $\Phi_{1\times1}=[1]$, $\mathrm{P}_{1\times1}=[\frac{1}{2}]$, and $\widehat{\Phi}_{m\times m}$is given by 5, while

$$\widehat{\Phi}^{-1}{}_{m\times m}=\frac{1}{m}\widehat{\Phi}^T{}_{m\times m}.\mathrm{diag}\left(1,1,2,2,\underbrace{2^2,\dots,2^2}_{2^2},\underbrace{2^3,\dots,2^3}_{2^3},\dots,\underbrace{\frac{m}{2},\dots,\frac{m}{2}}_{\frac{m}{2}}\right). \tag{8}$$

3. Numerical approximation of the solution

In this paper we have used the successive approximations method of (1), (2), with initial condition $u_0\in C[0,1]$,(usually $(f(x))$ .This iterative process will continue until a suitable error. Which usually occurs in 5^{th}or 6^{th} iteration. For any $x,t\in[0,1]$, and $n\geq 1$ and $m=2^{n+1}\in\mathbb{N}$, we define recursively

$$\psi_{n-1}(x,t):=K_1(x,t)W_1(t,u_{n-1})(t))\,, \tag{9}$$

$$\varphi_{n-1}(x,t):=K_2(x,t)W_2(t,u_{n-1}(t)). \tag{10}$$

If Q_m be an orthogonal projection with following interpolation property we have

$$Q_m(\psi)(x,t)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}s_{ij}^{(1)}h_i(x)h_j(t),$$

$$Q_m(\varphi)(x,t)=\sum_{i=0}^{m-1}\sum_{j=0}^{m-1}s_{ij}^{(2)}h_i(x)h_j(t).$$

Or in the matrix form

$$Q_m(\psi)(x,t)=h^T(x)S^{(1)}h(t), \tag{11}$$

$$Q_m(\varphi)(x,t)=h^T(x)S^{(2)}h(t). \tag{12}$$

Note that wherein $S^{(k)}=\left[s_{lq}^{(k)}\right]_{m\times m}$,for $k=1,2$

$$s_{lq}^{(1)}=2^{\frac{i+j}{2}}\langle h_{l(x)},\langle\psi(x,t),h_q(t)\rangle\rangle\,, \tag{13}$$

$$s_{lq}^{(2)}=2^{\frac{i+j}{2}}\langle h_l(x),\langle\varphi(x,t),h_q(t)\rangle\rangle\,, \tag{14}$$

with $i,j=0,1,\ldots,$where

$$l=2^j+kk=0,1,\ldots,2^j-1,$$
$$q=2^i+k'k'=0,1,\ldots,2^i-1,$$

and

$$h(t)=[h_0(t),h_1(t),\ldots,h_{m-1}(t)].^T \tag{15}$$

Now, by using the RH function vector $h(t)$, the matrix$\widehat{\Phi}_{m\times m}$is defined as:

$$\widehat{\Phi}_{m\times m}=\left[h\left(\frac{1}{2m}\right),h\left(\frac{3}{2m}\right),\ldots,h\left(\frac{2m-1}{2m}\right)\right]. \tag{16}$$

For example, the first four RH functions can be written in the matrix form as

$$\widehat{\Phi}_{m\times m}=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

Thus by using this equation we have

$$(S)^{(k)}=\left(\widehat{\Phi}^{-1}_{m\times m}\right)^T.\left(\hat{S}\right)^k.\ \left(\widehat{\Phi}\right)^{-1}_{m\times m},\quad k=1,2, \tag{17}$$

where $\left(\widehat{\boldsymbol{S}}\right)^k=\left[\hat{s}^{(k)}{}_{ij}\right]_{m\times m}$,for $k=1,2$, and $i,j=1,2,\ldots,m$ as

$$(\hat{s})^{(1)}{}_{ij}=\psi\left(\frac{2i-1}{2m},\frac{2j-1}{2m}\right)i,j=1,2,\ldots,m, \tag{18}$$
$$(\hat{s})^{(2)}{}_{ij}=\varphi\left(\frac{2i-1}{2m},\frac{2j-1}{2m}\right)i,j=1,2,\ldots,m \tag{19}$$

Thus for the Fredholm Hammerstein integral equations we have

$$u_n(x):=f(x)+\alpha\int_0^1 Q_m(\psi_{n-1})(x,t)dt,\quad n=1,2,\ldots, \tag{20}$$

for the Volterra - Hammerstein integral equations we have:

$$u_n(x):=f(x)+\beta\int_0^x Q_m(\varphi_{n-1})(x,t)dt,\quad n=1,2,\ldots. \tag{21}$$

4. Error analysis

In this section, by using the Banach fixed point theorem, we get an upper bound for the error of the our method, and the order of convergence is analyzed.

Lemma 1. Let $W_1,W_2:[0,1]^2\times\mathbb{R}\to\mathbb{R}$, be continuous and Lipschitzian with Lipschitz constant L_1 and L_2, and $\mathrm{K}_1,\mathrm{K}_2\in C([0,1]^2)$, be continiouse and$\mathrm{M}_1,\mathrm{M}_2\geq 0$ such that $|\mathrm{K_i}(\mathrm{x},\mathrm{t})|\leq \mathrm{M_i}$ for $i=1,2$, then T has an unique fixed point and for all $u_0\in C([0,1])$

$$\left\|u-T^i(u_0)\right\|_\infty\leq\|T(u_0)-u_0\|_\infty\times\sum_{j=i}^{\infty}q^j, \tag{22}$$

where $q=\ |\alpha|M_1L_1<1$, or $q=|\beta|M_2L_2<1$ and u is the fixed point of T.

Proof: For the Fredholm Hammerstein integral equations if $y,z\in C([0,1])$, we have:

$$
\begin{aligned}
&| T\big(y(x)\big) - T\big(z(x)\big)| \\
&\qquad = | \alpha \int_0^1 \Big(K_1(x,t)W_1\big(x,t,y(t)\big) - K_1(x,t)W_1\big(x,t,z(t)\big)\Big)\, dt \ | \\
&\qquad \le |\alpha| \int_0^1 |K_1(x,t)||W_1(x,t,y(t)) - W_1(x,t,z(t))| dt \\
&\qquad \le M_1 L_1 |\alpha| \int_0^1 |y(t) - z(t)| dt \\
&\qquad \le M_1 L_1 |\alpha| \|y - z\|_\infty,
\end{aligned}
$$

and for the Volterra Hammerstein integral equations if $y, z \in C([0,1])$, we have:

$$
\begin{aligned}
&\big|T\big(y(x)\big) - T\big(z(x)\big)\big| \\
&\qquad = \left|\beta \int_0^x \Big(|K_2(x,t)W_2\big(x,t,y(t)\big) - |K_2(x,t)W_2\big(x,t,z(t)\big)\Big)\, dt\right| \\
&\qquad \le |\beta| \int_0^x |K_2(x,t)|\big|W_2\big(x,t,y(t)\big) - W_2\big(x,t,z(t)\big)\big| dt \\
&\qquad \le M_2 L_2 |\beta| \int_0^x |y(t) - z(t)| dt \le M_2 L_2 |\beta| \|y - z\|_\infty.
\end{aligned}
$$

By induction, for the Fredholm - Volterra Hammerstein integral equations and every $n \in \mathbb{N}$ we have

$$|T^n(y) - T^n(z)|_\infty \le q^n |y - z|_\infty,$$

since $q < 1$ thus we have:

$$\sum_{n=1}^{\infty} \|T^n(y) - T^n(z)\|_\infty < \infty.$$

Thus T has a unique fixed point which means that (3),(4) has a unique solution and (22) follows from the Banach fixed-point theorem.

Theorem 1. Let $W_1, W_2 \colon [0,1]^2 \times \mathbb{R} \to \mathbb{R}$, be continuous and Lipschitzian with Lipschitz constants L_1 and L_2, $\mathrm{K}_1, \mathrm{K}_2 \in \mathrm{C}([0,1]^2)$, be continuous and $\mathrm{M}_1, \mathrm{M}_2 \ge 0$ such that $|\mathrm{K_i(x,t)}| \le \mathrm{M_i}$ for $i = 1,2$, then T has an unique fixed point and for all $u_0 \in C([0,1])$

$$\left\|u - T^i(u_0)\right\|_\infty \le \|T(u_0) - u_0\|_\infty \times \sum_{j=i}^{\infty} q^j, \tag{23}$$

where $q = |\alpha| M_1 L_1 < 1$, or $q = |\beta| M_2 L_2 < 1$ and u is the fixed point of T.

Proof. If $L_{i-1} = \max\{ \left\|\frac{\partial \psi_{i-1}}{\partial t}\right\|_\infty, \left\|\frac{\partial \psi_{i-1}}{\partial s}\right\|_\infty\}$, for the Fredholm Hammerstein integral equations, or $L_{i-1} = \max\{ \left\|\frac{\partial \varphi_{i-1}}{\partial t}\right\|_\infty, \left\|\frac{\partial \varphi_{i-1}}{\partial s}\right\|_\infty\}$for the Volterra Hammerstein integral equationsand $m = 2^{i+1}$ such that $i = 0,1,\ldots$,then

$$
\begin{aligned}
\|T(u_{i-1}) - u_i\|_\infty &\le |\alpha| \left\|\int_0^1 \psi_{i-1}(t,x) - Q_m(\psi_{i-1})(t,x) dt\right\|_\infty \\
&\le |\alpha| \|\psi_{i-1} - Q_m(\psi_{i-1})\|_\infty,
\end{aligned}
$$

for Fredholm Hammerstein integral equations and for Volterra Hammerstein integral equations we have

$$\|T(u_{i-1}) - u_i\|_\infty \le |\beta| \left\| \int_0^x \varphi_{i-1}(t,x) - Q_m(\varphi_{i-1})(t,x)dt \right\|_\infty \le |\beta| \|\varphi_{i-1} - Q_m(\varphi_{i-1})\|_\infty .$$

If we define $g(t,s) := \psi_{i-1} - Q_m(\psi_{i-1})$, and interpolate property and the mean-value theoremfor two variables with $t_0 = 0$, and

$$t_i = \frac{1}{2^{n_1+1}} + \frac{v_1}{2^{n_1}},$$

$$s_j = \frac{1}{2^{n_2+1}} + \frac{v_2}{2^{n_2}},$$

where $i = 2^{n_1} + v_1, j = 2^{n_2} + v_2, n_1, n_2 \ge 1, i, j \le m-1$, we have

$$\|\psi_{i-1} - Q_m(\psi_{i-1})\|_\infty = \left\| g(t_i, s_j) + \frac{\partial g}{\partial t}(\xi,\gamma)(\xi - t_i) + \frac{\partial g}{\partial t}(\xi,\gamma)(\gamma - s_j) \right\|_\infty$$

$$= \left\| (I - Q_m)\frac{\partial \psi_{i-1}}{\partial t}(\xi,\gamma) + (I - Q_m)\frac{\partial \psi_{i-1}}{\partial s}(\xi,\gamma) \right\|_\infty \max\{\|\xi - t_i\|_\infty, \|\gamma - s_j\|_\infty\}$$

$$\le \frac{2}{2^i} \|(I - Q_m)\|_\infty \left\| \frac{\partial \psi_{i-1}}{\partial t}(\xi,\gamma) + \frac{\partial \psi_{i-1}}{\partial s}(\xi,\gamma) \right\|_\infty,$$

similarly it holds for φ_{i-1} thus we have

$$\|T(u_{i-1}) - u_i\|_\infty \le |\alpha| \frac{4L_{i-1}}{2^i}, \tag{24}$$

or

$$\|T(u_{i-1}) - u_i\|_\infty \le |\beta| \frac{4L_{i-1}}{2^i}. \tag{25}$$

If $|\alpha| \frac{4L_{k-1}}{2^k} < \varepsilon_k$or $|\beta| \frac{4L_{k-1}}{2^k} < \varepsilon_k$, for$k = 1,2,\dots,i$, that $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_i > 0$ for $i \ge 1$, we have

$$\|T(u_{i-1}) - u_i\|_\infty < \varepsilon_i \tag{26}$$

Applying the triangle inequality and we achieve w

$$\|u - u_i\|_\infty \le |u - T^i(u_0)|_\infty + \sum_{j=1}^{i} q^{i-j} \left\| T(u_{j-1}) - u_j \right\|_\infty . \tag{27}$$

From (22) and (27) we conclude

$$\|u - u_i\|_\infty \le \|T(u_0) - u_0\|_\infty \sum_{j=i}^{\infty} q^j + \sum_{j=1}^{i} q^{i-j} \varepsilon_j . \tag{28}$$

Since $\varepsilon_j \to 0$ for $1 \le j \le i-1$, and $|q| < 1$ and

$$\sum_{j=i}^{\infty} q^j = \frac{q^i}{1-q},$$

then

$$\| u - u_i \|_\infty \le \alpha q^i,$$

which

$$\alpha = \frac{\|T(u_0) - u_0\|_\infty}{1-q},$$

thus the order of convergence is $O(q^i)$.

It should be noted that some of the following theorems and lemmas in this section has been proved for other bases, but we proved for rationalized Haar wavelets in a different way.

5. Numerical examples

In this section by using the method presented in (20), (21) are solved some examples from different references. The main characteristic of this technique is that does not lead to a nonlinear algebraic equations system. The following algorithm, based on the method presented in Section3, has been used to solve Examples 1 to 3.

Algorithm 1

1. Produce matrices $h(t), D_{m\times m}, andP$.
2. For $i = 1$to k do,
3. Product Matrix $S^{(k)}, \hat{S}^{(k)}$, for $k = 1,2$ from(13), (14), (17) and (18).
4. Compute $Q_m(\psi)(x,t)$ or $Q_m(\varphi)(x,t)$ from (11) or (12).
5. Compute $u_i(t)$ from (20) or (20) for the assumed point.
6. Go to step 2.

Example 1. Let us consider the nonlinear Fredholm Hammerstein integral equation of the second kind

$$u(t) = f(t) + \int_0^1 \left(\frac{sin(s-t)}{3}\right) u^2(s)ds\ , \quad (t \in [0,1]), \tag{29}$$

where for $t \in [0.1]$

$$f(t) = \frac{1}{9}\left(8cos(t) + 2sin(1)sin(t) + cos(1)^3 cos(t) + cos(1)^2 sin(1)sin(t)\right), \tag{30}$$

with $u(0) = 0$, and $K(t,s) = \frac{1}{3}sin(s-t)$,$W(s,u(s)) = u^2(s)$. The exact solution is $u(t) = cos(t)$.

The comparison between the approximate solutions obtained byHaar wavelet method and Schauder bases method [2] is given in Table1. There is a good agreement between these methods. In Fig. 1, we have showna comparison between the approximate solution with the exact solution. Inthis example the run time for $m = 128$ is about 0.612 seconds.

Table 1**.** Numerical results for Example 1

t_i	Schauder bases [2] With $j = 33$	Presented method With $m = 2^6$	Presented method With $m = 2^7$
0.24609375	2.64×10^{-7}	8.91×10^{-4}	3.73×10^{-6}
0.37109375	2.72×10^{-6}	9.15×10^{-4}	3.03×10^{-6}
0.49609375	2.76×10^{-6}	9.25×10^{-4}	2.28×10^{-6}
0.62109375	8.76×10^{-5}	9.20×10^{-4}	1.49×10^{-6}
0.74609375	2.72×10^{-5}	9.02×10^{-4}	6.87×10^{-7}
0.87109375	2.63×10^{-5}	8.69×10^{-4}	1.33×10^{-7}
0.99609375	2.50×10^{-5}	8.22×10^{-4}	9.52×10^{-7}

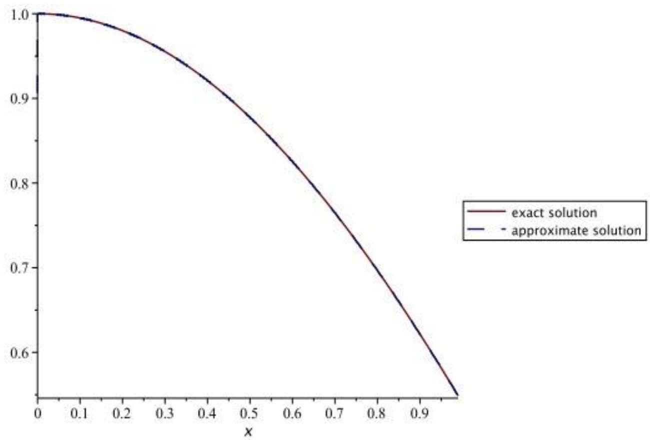

Figure 1: Comparison of exact and approximate solutions for Example 1

Example 2. Let us consider the nonlinear Fredholm integral equation of the second kind

$$u(t) = e^t + \frac{1}{4}e^{t-1} - \int_0^1 \frac{1}{4}e^{t-3s-1}u^3(s)ds, t \in [0,1], \qquad (31)$$

where $u(0) = 0$ and $K(t,s) = \frac{1}{4}e^{t-3s-1}$, $W(s,u(s)) = u^3(s)$ and exact solution is $u(t) = e^t$. In Table 2, the absolute error in the node $t_i \in [0,1]$ is shown. In the Table below, the exact solution u, byanswering repetitive u_i, can be approximated. Furthermore, the number used to represent the four pillars of m is also shown. In Fig.2, a comparison between analytical and approximate solutions is shown with a total run time about 2.078 seconds.

Table 2.Numerical results for Example 2

t_i	Our method With $m = 2^5$	Our method With $m = 2^6$	Our method With $m = 2^7$
0.24609375	1.84×10^{-3}	2.27×10^{-4}	4.93×10^{-5}
0.37109375	2.09×10^{-3}	2.57×10^{-4}	5.58×10^{-5}
0.49609375	2.37×10^{-3}	2.91×10^{-4}	6.32×10^{-5}
0.62109375	2.69×10^{-3}	3.30×10^{-4}	7.17×10^{-5}
0.74609375	3.04×10^{-3}	3.74×10^{-4}	8.13×10^{-5}
0.87109375	3.45×10^{-3}	4.24×10^{-4}	9.21×10^{-5}
0.99609375	3.91×10^{-3}	4.81×10^{-4}	1.04×10^{-4}
Run time (s)	0.063	0.360	1.5

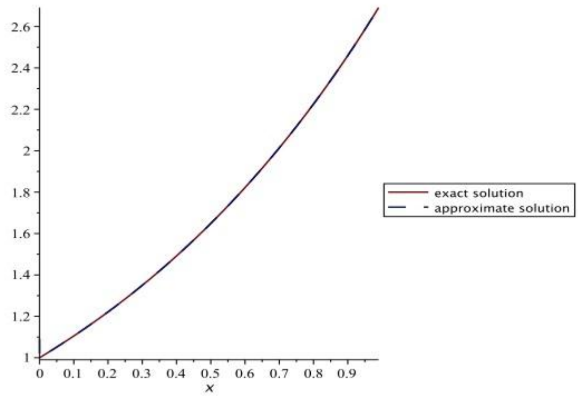

Figure 2: Comparison of exact and approximate solutions for Example 2

Example 3. Let us consider the nonlinear Volterra Hammerstein integral equation of the second kind

$$u(t) = f(t) + \int_0^t \frac{s^3 t^2}{2} arctan\big(u(s)\big)ds, \tag{32}$$

where

$$f(t) = t - t^2(\frac{1}{8}t^4 arctan(t) - \frac{1}{24}t^3 + \frac{1}{8}t - \frac{1}{8}arctan(t)),$$

and $u(0) = 0, W\big(s, u(s)\big) = s^2 arctan\big(u(s)\big)$, $K(t,s) = \frac{t^2}{2}s$, and the exact solution is $u(t) = t$.

The comparison between the approximated solutions obtained bythe Haar wavelet method, collocation-type method [8] and Legendrewavelets method [3] is given in Table 4. We see that there is agood agreement of results between these three methods. This fact justifiesthe ability, efficiency and applicability of the present method. In Fig. 3, comparison between analytical and approximated solutions is shown. The runtime for $m = 128$ is about 11.797 seconds.

Table 3. Numerical results for Example 3

t_i	collocation-type method[8] With $N = 65$	Legendre wavelets method[3] With $k = 1, M = 6$	Presented method With $m = 2^7$
0.12109375	1.68×10^{-6}	6.90×10^{-9}	3.10×10^{-10}
0.24609375	1.44×10^{-5}	5.36×10^{-7}	9.01×10^{-9}
0.37109375	1.78×10^{-5}	6.39×10^{-6}	6.72×10^{-8}
0.49609375	1.02×10^{-4}	3.61×10^{-5}	2.75×10^{-7}
0.62109375	3.90×10^{-4}	1.35×10^{-5}	8.14×10^{-7}
0.74609375	1.14×10^{-3}	3.96×10^{-6}	1.98×10^{-6}
0.87109375	2.81×10^{-3}	9.67×10^{-6}	4.31×10^{-6}
0.99609375	6.07×10^{-3}	2.07×10^{-5}	8.82×10^{-6}

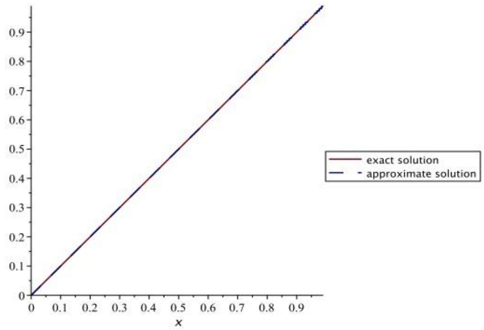

Figure 3: Comparison of exact and approximate solutions for Example 5.4

6. Conclusions

In this paper, we have used a numerical method which approximated the solution of the nonlinear Fredholm-Hammerstein integral equation (1) and nonlinear Volterra-Hammerstein integral equation (2)based on the expansion of the solution as series of Haar functions. These methods are not require the solutions of algebraic systems, we used the successive method for approximations of (1) and (2).

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Xətanın təhlili ilə qeyri-xətti Volter-Fredholm - Hammerşteyn inteqral tənliklərinin təqribi həlli üçün RH veyvlet bazis üsulu

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XÜLASƏ

Bu məqalədə Volter-Fredholm-Hammerşteyn inteqral tənliklərinin ədədi approksimasiyasının hesablanması üçün rasional Haar veyvletlərin xassələrindən istifadə edən üsul verilmişdir . Xətaların təhlili üçün əsas vasitə tərpənməz nöqtə haqqındakı Banax teoremidir. Xəta üçün yuxarı sərhəd tapılmışdır və yığılmanın tərtibi təhlil olunur. Bəzi ədədi misallar üçün həllin hesablanması alqoritmi verilmişdir.

Açar sözlər: qeyri-xətti inteqral tənlik; Rasionallaşdırılmış Haar veyvleti; Əməli matris; tərpənməz nöqtə haqqında teoremlər; xətaların təhlili

РХ вейвлетбазис метод приближенного решения нелинейных интегральных уравнений Вольтерра – Фредгольма -Гаммерштейна с анализом ошибок

Меджид Эрфаниан, Муртуза Гачпазан, Гусейн Беигло

РЕЗЮМЕ

В этой статье представлен метод для расчетной численной аппроксимации нелинейных фредгольмовых интегральных уравнений Вольтерра Хаммерштейна, который использует свойства рационализированных вейвлетов Хаара. Основным инструментом для анализа ошибок является теорема Банаха о неподвижной точке. Верхняя граница для ошибки был получен и порядок сходимости анализируется. Представлены алгоритм вычисления и иллюстирации решения для некоторых числовых примеров.

Ключевые слова: нелинейное интегральное уравнение, рационализированный вейвлет Хаара, оперативная матрица, теоремы о неподвижной точке, анализ ошибок.